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Bounds for Global Solutions of Some Semilinear Parabolic Equations

By

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1. Introduction. In this note we are concerned with the asymptotic behavior of global solutions of the initial boundary value problem $(E)_N$ (or $(E)_D$) for the semilinear parabolic equation :

$$(E) \quad \begin{cases} (1) & u_t(x,t) - \Delta u(x,t) = f(x,u(x,t)), \quad (x,t) \in \Omega \times [0,\infty), \\ (2) & u(x,0) = u_0(x), \quad x \in \Omega, \end{cases}$$

with the Neumann boundary condition

$$(3) \quad \partial u(x,t)/\partial n = 0, \quad (x,t) \in \partial\Omega \times [0,\infty),$$

(or the Dirichlet boundary condition

$$(4) \quad u(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,\infty).)$$

Here Ω is a bounded domain in \mathbb{R}^N with smooth boundary $\partial\Omega$ and f is a continuous function from $\Omega \times \mathbb{R}^1$ to \mathbb{R}^1 . It is well known that if f is superlinear in u , then (E) has solutions which blow up in finite time. So it would be natural to ask whether (E) has a global solution which blows up at ∞ or not.

This kind of problem was first studied by [7,8] for an abstract equation of the form $u_t + \partial\varphi^1(u(t)) - \partial\varphi^2(u(t)) = 0$ in a real Hilbert space \hat{H} , where $\partial\varphi^i$ are subdifferentials of lower semi-continuous convex and homogeneous functionals φ^i ($i=1,2$) on H . As an application of a result of [7], it is shown in [8] that

every global solution of $(E)_D$ with $f(x,u) = |u|^{p-2}u$ is uniformly bounded in $H_0^1(\Omega)$ with respect to time t , provided that $p < 2^*$, $2^* = \infty$ for $N=1,2$; $2^* = 2N/(N-2)$ for $N \geq 3$.

Ni-Sacks-Tavantzis [6] also studied $(E)_D$ for the case where Ω is convex and $f(x,u) = |u|^{p-2}u$ and showed that if $2 < p < 2 + 2/N$, then every positive global solution of $(E)_D$ is uniformly bounded in $L^\infty(\Omega)$ with respect to t and that if $2^* \leq p$ ($N \geq 3$), then $(E)_D$ has a global solution u such that $|u(t)|_{L^\infty} \rightarrow \infty$ as $t \rightarrow +\infty$.

Cazenave-Lions [2] treated more general nonlinear terms $f(x,u) = f_0(u)$ satisfying

$$(f)_0 \begin{cases} (o) & f_0 \in C^1(\mathbb{R}^1; \mathbb{R}^1), \\ (i) & |f_0(u)| \leq C_1 |u| + C_2 |u|^{p-1} \quad \forall u \in \mathbb{R}^1, \quad p < 2^*, \\ (ii) & u f_0(u) \geq (2+\varepsilon) F_0(u), \quad \forall u \in \mathbb{R}^1, \quad \varepsilon > 0, \quad F_0(u) = \int_0^u f_0(t) dt, \end{cases}$$

and showed that every global solution of $(E)_D$ is bounded in $L^\infty(\Omega)$ uniformly in $[t, \infty)$ for any $t > 0$. Furthermore this bound depends only on t and the $H_0^1(\Omega)$ -norm of u_0 , provided that $2 < p < 2_*$, $2_* = \infty$ for $N = 1$; $2_* = 2 + 12/(3N - 4)$ for $N \geq 2$. Giga [4] removed this restriction on p for positive solutions, i.e., he showed that if $p < 2^*$, then the $L^\infty(\Omega)$ bound for every positive global solution depends only on the $L^\infty(\Omega)$ -norm of u_0 .

For all these studies, it seems that there is no result for the Neumann problem $(E)_N$ in this direction in spite of its importance. In studying $(E)_N$, it must be noted that the methods in [2,4,6] rely much on the Dirichet boundary condition, so do not work for $(E)_N$.

The main purpose of this note is to show that "phase plane" method introduced in [7,8] works also for $(E)_N$. However, in order to apply this method to $(E)_N$, we have to remove two major restrictions in [7,8], i.e., conditions that $\partial\varphi^1$ is coercive and $\partial\varphi^2$ is homogeneous. As a matter of course, in carrying out this, we need much more careful consideration in the phase space than in [7,8]. In this note we are concerned with the following nonlinearity of f :

$$(f) \quad \left\{ \begin{array}{l} f(.,.) \text{ is a continuous function from } \Omega \times \mathbb{R}^1 \text{ into } \mathbb{R}^1 \\ \text{and there exist constants } K_i (i=0,1,2,3) \text{ and numbers} \\ p \in (2, 2^*), \delta > 0 \text{ and } \xi > 0 \text{ such that} \\ (i) \quad |f(x,u)| \leq K_0 (1 + |u|^{p-1}) \quad \forall (x,u) \in \Omega \times \mathbb{R}^1, \\ (ii) \quad F(x,u) = \int_0^u f(x,t) dt \geq K_1 |u|^{2+\delta} - K_2 \quad \forall (x,u) \in \Omega \times \mathbb{R}^1, \\ (iii) \quad u f(x,u) \geq (2 + \xi) F(x,u) - K_3 \quad \forall (x,u) \in \Omega \times \mathbb{R}^1. \end{array} \right.$$

Then our main results are stated as follows.

Theorem I. Let (f) be satisfied and u be a global solution of $(E)_N$ (or $(E)_D$) such that $u \in V \equiv W_{loc}^{1,2}([0, \infty); L^2(\Omega)) \cap L_{loc}^2([0, \infty); H^2(\Omega))$. Then there exists a positive constant $C_0 = C_0(|u_0|_{H^1}, K_0, K_1, K_2, K_3, \delta, \xi, |\Omega|)$ such that

$$(5) \quad \sup_{t \geq 0} \|u(t)\|_{L^2} \leq C_0,$$

$$(6) \quad \sup_{t \geq 0} \|u(t)\|_{H^1} < +\infty,$$

$$(7) \quad \text{There exists a number } T_1 \text{ such that } \sup_{t \geq T_1} \|u(t)\|_{H^1} \leq C_0,$$

$$(8) \quad \sup_{t \geq 0} \|u(t)\|_{H^1} \leq C_0, \text{ provided that } p \in (2, 2_*), 2_* = \infty \text{ for } N=1 \\ \text{and } 2_* = 2 + 12/(3N-4).$$

Theorem II. Let (f) be satisfied and u be a global solution of $(E)_N$ (or $(E)_D$) such that $u_0 \in L^\infty(\Omega)$ and $u \in L_{loc}([0, \infty); L^\infty(\Omega)) \cap W_{loc}^{1,2}((0, \infty); L^2(\Omega)) \cap L_{loc}^2((0, \infty); H^2(\Omega))$. Then there exists a positive constant $C_1 = C_1(|u_0|_{L^\infty}, K_0, K_1, K_2, K_3, \delta, \varepsilon, |\Omega|)$ such that

$$(9) \quad \sup_{t \geq 0} |u(t)|_{L^\infty} < \infty,$$

$$(10) \quad \text{There exists a number } T_1 \text{ such that } \sup_{t \geq T_1} |u(t)|_{L^\infty} \leq C_1,$$

$$(11) \quad \sup_{t \geq 0} |u(t)|_{L^\infty} \leq C_1, \text{ provided that } p \in (2, 2_*).$$

Remark 1. (1) Assertions in Theorems I and II hold true also for Robin problem $(E)_R$, i.e., (E) with more general boundary condition:

$$(3)' \quad \frac{\partial u}{\partial n}(x, t) + \sigma(x) u(x, t) = 0, (x, t) \in \partial\Omega \times [0, \infty), \sigma \in L^\infty(\partial\Omega).$$

The following arguments for $(E)_N$ can be applied for $(E)_R$ with slight modifications.

(2) Conditions $(f)_0$ does not allow f to contain linear or sublinear parts, but condition (f) allow it. For example, $f(x, u) = |u|^{q-2}u + |u|^{p-2}u$ with $1 < q \leq 2 < p$ satisfies (f) but not $(f)_0$.

(3) If f satisfies condition (f) , then so does $g(x, u) = f(x, u) + u$. Indeed it is clear that g satisfies (i)-(ii) of (f) , and since

$$u g(x, u) = u^2 + u f(x, u) \geq u^2 + (2 + \varepsilon) F(x, u) - K_3,$$

$$(2 + \varepsilon/2)(u^2/2 + F(x, u)) + \varepsilon F(x, u)/2 - \varepsilon u^2/4 - K_3,$$

$$(2 + \varepsilon/2) \int_0^u g(x, t) dt + \varepsilon(K_1 |u|^{2+\delta} - K_2 - u^2/2)/2 - K_3,$$

and $K_1 |u|^{2+\delta} - K_2 - |u|^2/2$ is bounded below, $g(x,u)$ also satisfies (iii) of (f) with ε and K_3 replaced by $\varepsilon/2$ and some K'_3 .

2. Proofs of Theorems. We shall give here proofs only for Neumann problem $(E)_N$, which are more complicated than those for $(E)_D$ and also valid for $(E)_D$.

Instead of (1), let us here consider its equivalent:

$$(1)' \quad u_t - \Delta u + u = u + f(x,u) = g(x,u) \quad (x,t) \in \Omega \times [0,\infty).$$

As was seen in (3) of Remark 1, g also satisfies (f). We use the same K_1 , δ and ε for g as before. In what follows, (\cdot, \cdot) and $|\cdot|$ denote the inner product and norm of $L^2(\Omega)$. We also denote by $|\cdot|_r$ and $\|\cdot\|$ the $L^r(\Omega)$ norm and $H^1(\Omega)$ norm respectively. Now we introduce several functionals on $H^1(\Omega)$

$$- \quad A(u) = \frac{1}{2} (|\nabla u|^2 + |u|^2) = \frac{1}{2} \|u\|^2$$

$$- \quad G(u) = \int_{\Omega} \int_0^{u(x)} g(x,t) dt dx,$$

$$- \quad J(u) = A(u) - G(u),$$

$$- \quad j(u) = (g(\cdot, u(\cdot)), u(\cdot)) - 2A(u)$$

and subsets of $H^1(\Omega)$:

$$- \quad S_a = \{ u \in H^1(\Omega) ; j(u) = a \}, \quad a \in \mathbb{R}^1.$$

Then, by virtue of (ii) and (iii) of (f), we obtain

$$(12) \quad G(u) \geq K_1 |u|_{2+\delta}^{2+\delta} - K_2 |\Omega| \quad \forall u \in H^1(\Omega), (|\Omega| \text{ is the volume of } \Omega.)$$

$$(13) \quad j(u) \geq \varepsilon G(u) - 2J(u) - K_3 \quad \forall u \in H^1(\Omega),$$

$$(14) \quad A(u) \geq (1 + \varepsilon/2) G(u) - K_3/2 - a/2 \quad \forall u \in S_a.$$

Furthermore, by (i) of (f) and Sobolev's inequality :

$|u|_p \leq C_b \|u\|$, there exists a constant d_0 such that $d_0 \geq K_3$ and $G(u) \leq d_0 (A(u)^{p/2} + 1)$, whence follows

$$(15) \quad A(u) \geq ([G(u)/d_0 - 1]^+)^{2/p}, \quad \text{where } [a]^+ = \max(a, 0).$$

(Note that d_0 is a constant depending only on $K_0, K_3, p, |\Omega|$ and embedding constant C_b .) Introducing a new parameter $\alpha = (2 + \varepsilon) d_0 - K_3 > 0$ and taking account of (14) and (15), we can draw the following Fig. 1 which illustrates how S_0, S_α and lines $J(u) = J(u_0)$ and $J(u) = -d_0$ are located in the $(G(u), A(u))$ -phase plane.

We here claim the following proposition.

Proposition 1. Let u be a global solution of $(E)_N$ belonging to V . Then we have

(i) $J(u(t))$ is monotone decreasing in t ,

(ii) $J(u(t)) \geq -d_0 \quad \forall t \geq 0$,

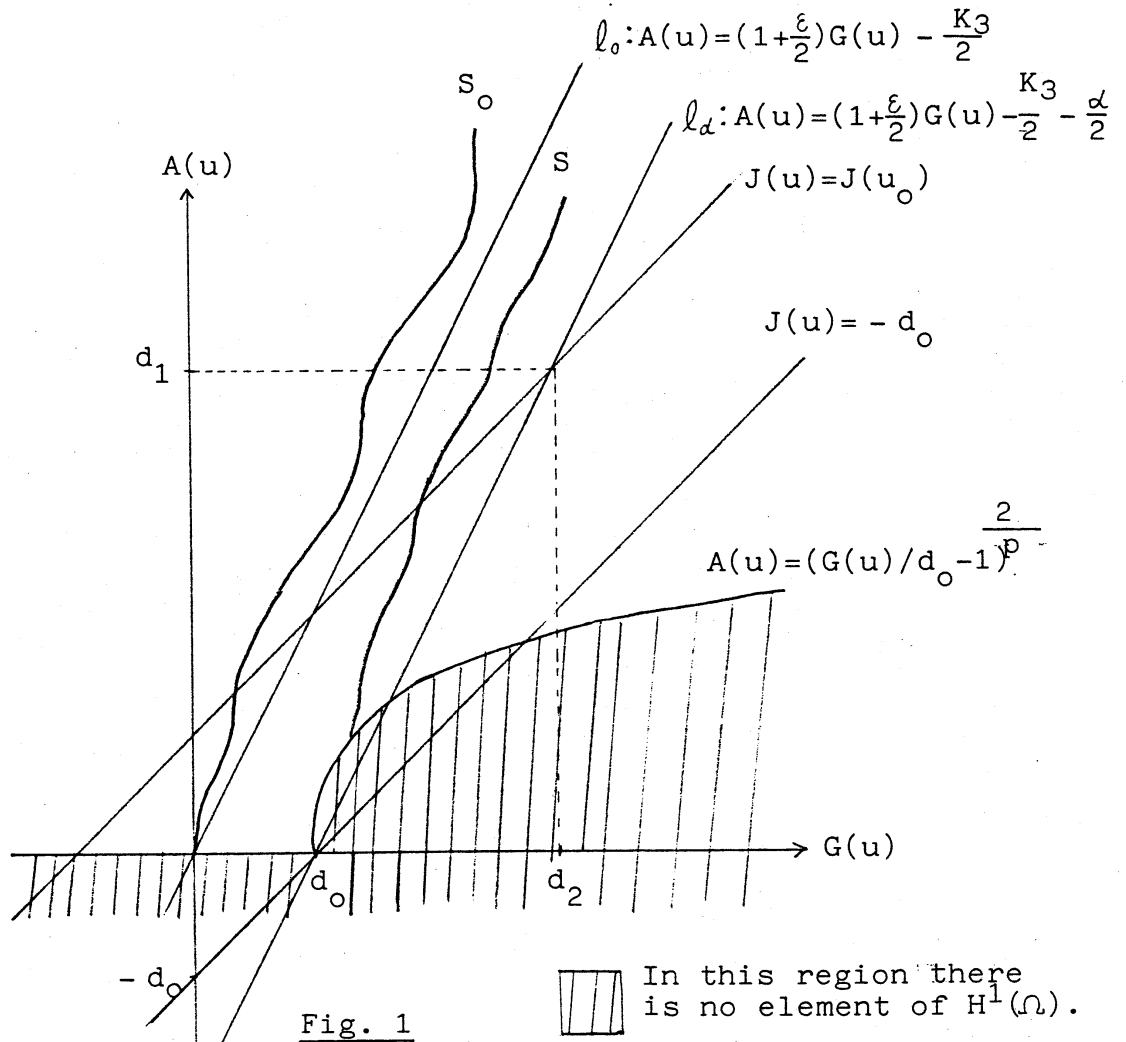
(iii) $\int_0^\infty |u_t(t)|^2 dt \leq J(u_0) + d_0$,

(iv) $|u(t)| \leq C_0 \quad \forall t \geq 0$,

(v) $\int_t^{t+1} G(u(s))^2 ds \leq C_0 \quad \forall t \geq 0$,

(vi) $\int_t^{t+1} \|u(s)\|^4 ds \leq C_0 \quad \forall t \geq 0$,

where C_0 is a constant depending only on K_i ($i=0,1,2,3$), ε, δ , d_0 and $J(u_0)$.



$$\alpha = (2 + \varepsilon) d_0 - K_3 > 0$$

$$d_1 = (2 + \varepsilon) (J(u_0) + d_0) / \varepsilon$$

$$d_2 = (2 J(u_0) + (2 + \varepsilon) d_0) / \varepsilon$$

$$d_0 = d_0(K_0, K_3, p, |\Omega|, C_b)$$

Proof of Proposition 1. Multiplying (1)' by u_t and u , we have

$$(16) \quad |u_t(t)|^2 + \frac{d}{dt} J(u(t)) = 0 \quad \text{for a.e. } t \in [0, \infty),$$

$$(17) \quad \frac{1}{2} \frac{d}{dt} |u(t)|^2 = j(u(t)) \quad \text{for a.e. } t \in [0, \infty).$$

Then assertion (i) is a direct consequence of (16), and it follows from (12), (13), (17) and (i) that

$$(18) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} |u(t)|^2 &\geq \varepsilon G(u(t)) - 2J(u_0) - K_3 \\ &\geq \varepsilon K_1 |\Omega|^{-\delta/2} |u(t)|^{2+\delta} - \varepsilon K_2 |\Omega| - 2J(u_0) - K_3. \end{aligned}$$

Suppose that $J(u(t_0)) < -d_0$ for some t_0 , then in view of (i) and Fig. 1, we find that $j(u(t)) \geq \alpha > 0$ for all $t \geq t_0$. Hence (17) and (18) assure that there exists $t_1 > 0$ such that

$$(19) \quad \frac{d}{dt} |u(t)|^2 \geq \varepsilon K_1 |\Omega|^{-\delta/2} |u(t)|^{2+\delta} \quad \text{for a.e. } t \geq t_1,$$

which implies that $|u(t)|$ blows up in a finite time. This is a contradiction. Thus (ii) is verified. Consequently integration of (16) over $[0, \infty)$ gives (iii). Suppose now that there exists a $t_1 > 0$ such that

$$\varepsilon K_1 |\Omega|^{-\delta/2} |u(t_1)|^{2+\delta} / 2 \geq \varepsilon K_2 |\Omega| + 2J(u_0) + K_3,$$

then by (18), $|u(t)|$ is monotone increasing in the neighbourhood of t_1 . Consequently (19) holds for a.e. $t \geq t_1$, this again leads to a contradiction. Thus we obtain a priori bound:

$$(20) \quad \sup_{t \geq 0} |u(t)| \leq K_4 \equiv \left[\{4 \varepsilon K_2 |\Omega| + 4J(u_0) + 2K_3\} |\Omega|^{\delta/2} / \varepsilon K_1 \right]^{1/(2+\delta)}$$

Since $d|u(t)|^2/dt = 2(u(t), u_t(t)) \leq 2|u(t)||u_t(t)|$, assertion (v) is derived from (iii), (iv) and integration of (18) over $[t, t+1]$. Assertion (vi) follows from (v) and the fact that $J(u(t)) \leq J(u_0)$ or $A(u(t)) \leq B(u(t)) + J(u_0)$. Q.E.D.

Before proceeding to the proof of Theorem I, we prepare the following lemmas.

Lemma 1. Let $r, q \in [1, \infty]$ and m be a non-negative integer.

Then

$$|u|_s \leq C |u|_{W^{m,r}}^a |u|_q^{1-a} \quad \forall u \in W^{m,r}(\Omega) \cap L^q(\Omega)$$

holds for any numbers $a \in [0, 1]$ and $s \in [1, \infty]$ satisfying

$$1/s = a(1/r - m/N) + (1-a)/q,$$

where C is a constant depending only on Ω, r, q, m and a .

For a proof of this lemma, see Friedmann [3].

Lemma 2. Let (i) of (f) be satisfied and u be a global solution of $(E)_N$ belonging to V . Then there exists a positive monotone decreasing function $T(\cdot)$ such that

$$(21) \quad \|u(t)\| \leq \|u(t_0)\| + 1 \quad \text{for all } t_0 \text{ and } t \in [t_0, t_0 + T(\|u(t_0)\|)].$$

Proof. First of all, we note that there exists a number $\lambda \in (0, 2]$ such that

$$(22) \quad |u|_{2(p-1)}^{2(p-1)} \leq C |u|_{H^2}^{2-\lambda} \|u\|^{2p-4+\lambda} \quad \forall u \in H^2(\Omega).$$

Indeed, for the case $N=1, 2$ or $N \geq 3$ and $2(p-1) \leq 2N/(N-2)$, we can take $\lambda = 2$. For the other case, we have only to apply Lemma 1 with $s = 2(p-1)$, $m = r = 2$ and $q = 2N/(N-2)$ and use the

fact that $H^1(\Omega)$ is continuously embedded in $L^{2N/(N-2)}(\Omega)$.

This inequality implies that there exists a monotone increasing function $M(\cdot)$ such that

$$(23) \quad |g(\cdot, u)|^2 \leq \frac{1}{2} |\Delta u|^2 + M(\|u\|) \quad \forall u \in H^2(\Omega).$$

Multiplying (1)' by $-\Delta u(t) + u(t)$, we get

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|^2 + |\Delta u(t)|^2 + |u(t)|^2 + 2 |\nabla u(t)|^2 \leq |g(\cdot, u)| |\Delta u(t)|.$$

Then, by (23), we obtain

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|^2 \leq M(\|u(t)\|) \quad \text{for a.e. } t \in [0, \infty),$$

by which we can easily verify (21) by taking $T(r) = 1/2M(r+1)$.

Q.E.D.

Now we proceed to the proof of Theorem I.

Proof of (6) and (7). By virtue of (iii) of Proposition 1, there exists a positive number T_0 such that

$$(24) \quad \int_{T_0}^{\infty} |u_t(t)|^2 dt \leq \left(\frac{\alpha}{K_4} \right)^2 T(d_1).$$

Then we have

$$(25) \quad \|u(t)\| \leq d_1 + 1 \quad \text{for all } t \geq T_1 \equiv T_0 + \frac{K_4^2}{2\alpha}.$$

Suppose that this does not hold. Then there exists a $t_1 \geq T_1$ such that $\|u(t_1)\| > d_1 + 1$. Hence, by Fig. 1, $j(u(t_1)) > \alpha$.

Therefore there exists a $t_0 < t_1$ such that $j(u(t_0)) = \alpha$ and $j(u(t)) > \alpha$ for all $t \in (t_0, t_1]$. Then integration of (17) over $[t_0, t_1]$ and (20) give $t_1 - t_0 \leq K_4^2 / 2\alpha$, i.e., $t_0 \geq T_0$.

Again integrating (17) over $[t_0, t_1]$, we find

$$\begin{aligned} \alpha(t_1 - t_0) &\leq \int_{t_0}^{t_1} j(u(t)) dt \leq \int_{t_0}^{t_1} |u_t(t)| |u(t)| dt \\ &\leq K_4 (t_1 - t_0)^{1/2} \left(\int_{t_0}^{t_1} |u_t(t)|^2 dt \right)^{1/2}. \end{aligned}$$

Hence, by (24), $t_1 - t_0 \leq T(d_1)$. Since $\|u(t_0)\| \leq d_1$, Lemma 2 assures that $\|u(t_1)\| \leq d_1 + 1$. This is a contradiction. Thus (5)-(7) of Theorem I are verified.

Proof of (8). As was seen above, $u(t)$ can not stay in the region $\{u \in H^1(\Omega); j(u) > \alpha\}$ longer than $\gamma \equiv K_4^2 / 2\alpha$. Then for any $t \geq 0$, there exists $t_0 \in [\max(0, t-\gamma), t]$ such that $\|u(t_0)\| \leq d_3 \equiv \max(\|u_0\|, d_1)$. Therefore, in order to prove (8), it suffices to show that $\|u(t)\| \leq C_0$ for all $t \in I \equiv [t_0, t_0 + \gamma]$. For this purpose, we prepare several results on u .

Lemma 3. If $u \in L^r(I; L^r(\Omega))$, $r < 2^*$, then

$$|u|_{L^\infty(I; L^{r/2+1}(\Omega))} \leq C(|u|_{L^r(I; L^r(\Omega))}, d_0, d_3, r).$$

Proof. We note that

$$\begin{aligned} \frac{1}{s} \frac{d}{dt} |u(t)|_s^s &= (u_t(t), |u|^{s-2} u(t)) \leq |u_t(t)| |u(t)|_r^{s-1} \\ &\leq \frac{1}{2} (|u_t(t)|^2 + |u(t)|_r^r), \quad s = \frac{r}{2} + 1. \end{aligned}$$

Then integration of this over I and (iii) of Proposition 1 assure the assertion. Q.E.D.

Proposition 2. If $u \in L^\infty(I; L^r(\Omega))$ with $N(p-2)/2 < r < 2^*$, then

$$|u|_{L^\infty(I; H^1(\Omega))} \leq C(|u_0|_{L^\infty(I; L^r(\Omega))}, C_0).$$

Proof. Multiplying (1)' by $|u|^{\ell-2}u(t)$ ($2 < \ell < 2^*$), we obtain, by (i) of (f),

$$(26) \quad \frac{1}{\ell} \frac{d}{dt} |u(t)|_{\ell}^{\ell} + \frac{4(\ell-1)}{\ell^2} \| |u(t)|^{\ell/2} \|^2 \leq K_0 (2 |u(t)|_{\frac{p+\ell-2}{p+\ell-2}}^{p+\ell-2} + |\Omega|).$$

For all $\ell > (p-2)N/2$, Lemma 1 with $s = 2(p+\ell-2)/\ell$, $m = 1$, $r = q = 2$ and u replaced by $|u|^{\ell/2}$ gives

$$(27) \quad |u|_{\frac{p+\ell-2}{p+\ell-2}}^{p+\ell-2} \leq C |u|^{p+\lambda\ell-2} \| |u|^{\ell/2} \|^{2-\lambda}, \quad \lambda > 0.$$

Then, by integration of (26) over I and (27) with $\ell = r$, we get

$$(28) \quad \| |u|^{r/2} \|_{L^2(I; H^1(\Omega))} \leq C (\|u\|_{L^\infty(I; L^r(\Omega))}^r, \|u_0\|)$$

Again by Lemma 1 with $s = 2(N+2)/N$, $m = 1$, $r = q = 2$ and u replaced by $|u|^{r/2}$, we now deduce

$$(29) \quad |u|_{\frac{(2+N)r/N}{(2+N)r/N}}^{(2+N)r/N} \leq C |u|_r^{2/N} \| |u|^{r/2} \|^2.$$

Then it follows from (28) and (29) that $u \in L^{(2+N)r/N}(I; L^{(2+N)r/N}(\Omega))$. Hence, by Lemma 3, $u \in L^\infty(I; L^{(N+2)r/2N+1}(\Omega))$. Repeating this procedure, we observe that $u \in L^\infty(I; L^{r_i}(\Omega))$, where r_i are defined by the recurrence formula

$$r_{i+1} = \frac{N+2}{2N} r_i + 1, \quad i = 1, 2, \dots, \quad r_1 = r.$$

Since $r_i \rightarrow 2^*$ as $i \rightarrow \infty$ and $p < 2^*$, we can show by finite steps that $|u|_{L^\infty(I; L^p(\Omega))} \leq C (\|u\|_{L^\infty(I; L^r(\Omega))}^r, C_0)$. Thus the assertion follows from (ii) of Proposition 1. Q.E.D.

Proposition 3. For $N=1,2,3$ or 4 , we have

$$|u|_{L^\infty(I; L^q(\Omega))} \leq C_0 \quad \text{for all } q < q^*,$$

where $q^* = \infty$ for $N=1$ and $q^* = 2 + 8/(3N-4)$ for $N=2,3,4$.

Proof. Let $q_1 = 2$ and $q_{i+1} = (4-N)q_i/2N + (6N+8)/2N$, $i = 1, 2, \dots$, and apply Lemma 1 with $s = q_{i+1}$, $m = 1$, $r = 2$ and $q = (2+q_i)/2$. Then we have

$$|u|_{q_{i+1}}^{q_{i+1}} \leq C |u|_{(2+q_i)/2}^{q_{i+1}-4} \|u\|^4.$$

Therefore, by virtue of (vi) of Proposition 1 and Lemma 3,

$u \in L^{q_i}(I; L^{q_i}(\Omega))$ for all q_i . Since $q_i \rightarrow q^*$ as $i \rightarrow \infty$, we can derive the assertion by finite steps. Q.E.D.

Proposition 4. For $N \geq 5$, we have

$$|u|_{L^\infty(I; L^q(\Omega))} \leq C_0 \quad \text{for all } q < q_* = 3 + (N-4)(2-p)/4.$$

Proof. Since $p < 2^* \leq 4$ for $N \geq 4$ and (vi) of Proposition 1 assures that $|u|_{L^4(I; L^{2^*}(\Omega))} \leq C_0$, we get $|u|_{L^p(I; L^p(\Omega))} \leq C_0$.

Let $p_1 = p$, $p_{i+1} = (N-4)p_i/(N-2) + [8+(N-4)(2-p)]/(N-2)$, and

$s_i = 2 + p_i - p$, $i = 1, 2, \dots$. Suppose that $u \in L^{p_i}(I; L^{p_i}(\Omega))$,

then by (26) with $\ell = s_i$, $|u|^{s_i/2} \in L^2(I; H^1(\Omega))$. Moreover

Lemma 1 with $s = 2p_{i+1}/s_i$, $m = 1$, $r = 2$, $q = 2 \cdot 2^*/s_i$ and u replaced by $|u|^{s_i/2}$ yields

$$|u|_{p_{i+1}}^{p_{i+1}} \leq C |u|_{2^*}^{\lambda_1} \| |u|^{s_i/2} \|^{\lambda_2} \quad \text{with } \lambda_1/2 + \lambda_2 \leq 2,$$

whence follows $|u|_{L^{p_{i+1}}(I; L^{p_{i+1}}(\Omega))} \leq C_0$. Since $p_i \rightarrow q_*$ as

$i \rightarrow \infty$, the assertion can be derived by finite steps. Q.E.D.

In order to prove (8), we have only to combine Propositions 2 with 3 (for the case $N \leq 4$) and 2 with 4 (for the case $N \geq 5$). In fact, $N(p-2)/2 < q^*$ or $N(p-2)/2 < q_*$ holds if and only if $p < 2_*$. Q.E.D.

Proof of Theorem II. We shall rely on Moser's iteration scheme to obtain L^∞ bound via H^1 bound. The following lemma plays an important role in this procedure.

Lemma 4. Let $w \in W_{loc}^{1,1}([0, \infty); L^2(\Omega)) \cap L_{loc}^\infty([0, \infty); L^\infty(\Omega) \cap H^1(\Omega))$

satisfy

$$(30) \quad \frac{d}{dt} |w(t)|_r^r + C_1 r^{-\theta_1} \| |w(t)|^{r/2} \|^2 \leq C_2 r^{\theta_2} (|w(t)|_r^{r+1}) \quad \text{a.e. } t \in [0, \infty)$$

for all $r \geq 2$, where $C_1 > 0$ and $C_2, \theta_1, \theta_2 \geq 0$.

Then there exist constants a, b, c, d such that

$$\sup_{t \geq 0} |w(t)|_\infty \leq a 2^{\theta_2 + (\theta_1 + \theta_2)b} M_0,$$

where $M_0 = \max(1, c|u_0|_\infty, \sup_{t \geq 0} |w(t)|^d)$.

Proof. When w belongs to $H_0^1(\Omega)$ for a.e. t , this is proved by Nakao [5] (See Lemma 3.1). By using Lemma 1 instead of Gagliardo-Nirenberg's inequality, one can prove this lemma by the same argument as in [5]. Q.E.D.

Put $\lambda_1 = r(1 - N(p-2)/2p)$, $\lambda_2 = p-2$, $\lambda_3 = Nr(p-2)/2p$ and $\theta = Nr/(N-2)$. Then, by Hölder's inequality, we get

$$|u|_{p+r-2}^{p+r-2} \leq |u|_r^{\lambda_1} |u|_p^{\lambda_2} |u|_\theta^{\lambda_3}.$$

Applying Sobolev's embedding theorem and Young's inequality, we deduce

$$\begin{aligned}
 (31) \quad |u(t)|_{p+r-2}^{p+r-2} &\leq C |u(t)|_r^{\lambda_1} \| |u(t)|^{r/2} \|^{2\lambda_3/r} \\
 &\leq \frac{2(r-1)}{r^2} \| |u(t)|^{r/2} \|^2 + C r^{\frac{N(p-2)}{2p-(p-2)N}} |u(t)|_r^r,
 \end{aligned}$$

where C is a general constant depending on $\sup\{|u(t)|_p; t \geq 0\}$. Then (31) and (26) with $\ell = r$ imply that $u(t)$ satisfies (30) with $C_1 = 1$, $\theta_1 = 0$ and $\theta_2 = 2p/(2p-(p-2)N)$. Thus (9) is verified by Lemma 4.

On the other hand, it is easy to show that there exists a positive number T_0 depending only on $|u_0|_\infty$ such that $|u(t)|_\infty \leq |u_0|_\infty + 1$ for all $t \in [0, 2T_0]$ and $\|u(T_0)\| \leq C(|u_0|_\infty)/T_0$. Hence, (10) and (11) follows from (7) and (8) respectively.

Q.E.D.

References

- 1 Alikakos, N.D., L^p bounds of solutions of reaction-diffusion equations, Comm. Partial Differential Equations 4 (1979), 827-868.
- 2 Cazenave, T. and Lions, P.L., Solutions globales d'équations de la chaleur semi linéaires, ibid., 9 (1984), 955-978.
- 3 Friedmann, A., Partial Differential Equations, Holt, Rinehart and Winston, New York (1969).
- 4 Giga, Y., A bound for global solutions of semilinear heat equations, Comm. Math. Phys., 103 (1986), 415-421.
- 5 Nakao, M., L^p -estimate of solutions of some nonlinear degenerate diffusion equations, J. Math. Soc. Japan, 37 (1985), 41-64.
- 6 Ni, W.M., Sacks, P.E., and J. Tavantzis, On the asymptotic behavior of solutions of certain quasilinear equation of parabolic type, J. Differ. Equations 54 (1984), 97-120.
- 7 Ôtani, M., Existence and asymptotic stability of strong solution of nonlinear evolution equations with a difference term of subdifferentials, Colloquia Math. Soc. Janos Bolyai, 30, Qualitative theory of differential equations, North Holland, 1980.
- 8 Ôtani, M., 寄微分作用素の差の項をもつ発展方程式の解の漸近挙動について, 京都大学数理解析研究所研究録 (1980)